International Journal of Theoretical Physics, Vol. 18, No. 11, 1979

Mathematical Theory of Reduction of Physical Parameters and Similarity Analysis

K. L. Chan and W. Y. Chau

Physics Department, Queen's University at Kingston, Ontario, Canada

Received December 8, 1978

The method of similarity analysis in the study of differential equations is extended to study the variability of parameters in a physical system. The analysis provides an insight into the meaning of "physical" similarity, which usually means the possibility of a reduction in the number of physical parameters characterizing the system. Theorems relating similarities to groups of invariant transformations are proved and employed to show how the number of parameters can be reduced.

1. INTRODUCTION

A physical system usually has many changeable attributes. When this physical system is modeled by a set of differential equations, these changeable attributes appear in the form of parameters. For example, in Reynold's modeling of incompressible viscous fluid flows around a submerged body, the equations

$$\nabla \times [(\mathbf{V}\cdot\nabla)\mathbf{V}] = \nu \nabla \times (\nabla^2 \mathbf{V})$$

$$\nabla \cdot \mathbf{V} = 0$$

together with the boundary conditions

$$\mathbf{V} = \begin{cases} U_0 \hat{x} & \text{at } x = -\infty \\ 0 & \text{at the object surface } S(\mathbf{x}) = 0 \end{cases}$$

determine one solution when the kinematic viscosity ν, upstream velocity U_0, and the surface shape of the object are *fixed*. These are physical attributes specifying a state for the physical system; they appear as parameters in the mathematical equations. When one says that flows with equal Reynold's number are similar to each other, one is actually not speaking about solutions to the equations for a single set of parameters —one is speaking about relations among solutions of *different* mathematical equations generated by different values of parameters. Therefore, "similarity" is not a concept involving only the differential equations, it is a relationship describing the behavior of solutions under change of parameters. In standard dimensional or similarity analysis, while much attention was paid towards reducing the number of independent variables for a simplification of the differential equations, the importance of the variability of the parameters was not pointed out explicitly. In this paper, we analyze the mathematical concept of "similarity" by taking into account the important role of parameters and discuss how similarity analysis can be generalized to a broader domain.

In Section 2, the nature of mathematical modeling of a physical system is studied; similarity of physical states is given a mathematical meaning. In Section 3, the group method is used to investigate similarity in the space of solutions. The meaning of reducing the number of parameters for a physical system is clarified. Theorems on the relationship between the group of invariant transformations (see later discussion) and similarity of solutions and on the number of parameters that can be reduced by such a group will be given. In Section 4, examples will be given to illustrate the concepts and application.

For conciseness and precision, mathematical notations and terminology will be heavily employed in our discussion.

2. MATHEMATICAL MODELING OF A PHYSICAL SYSTEM

We are interested in cases that the physical system can be modeled by a system of differential equations. When we say a "system," we assume that appropriate boundary conditions have been included so that the mathematical problem is well posed (unique solution exists) when values of the parameters are fixed. Parameters are those constants that do not change in a fixed state (yielding one solution), but may be changed and give rise to *different* solutions. Even though the solutions for different values of parameters are generally different, some of them may be obtained from each other by transformations involving the variables. When they are related like this, we say that they are *similar* to each other. When similarities exist among solutions, many related solutions can be found by a single integration of the equations. Even when solutions for a large *range* of parameters are wanted, only those for a subset of the *range* need to be calculated. Other solutions can be generated by known transformations. In this sense the parameters are *reducible*. How can we tell whether similarities exist? Similar to the reduction of independent variables in ordinary similarity analysis, parameters can be reduced when transformations preserving the form of the system of differential equations can be found. We are going to make these points more precise in later discussion.

Consider a system $\Sigma(p,x,y)$ of differential equations (including boundary conditions) with l parameters $p \in \mathsf{P}$ (range of parameters) $\subset \mathsf{R}^l$ ($\mathsf{R} \equiv$ the set of real numbers), m independent variables $x \in \mathsf{X}$ (domain of independent variables) $\subset \mathsf{R}^m$ and n dependent variables $y \in \mathsf{Y}$ (range of dependent variables) $\subset \mathsf{R}^n$. We shall write $\mathsf{P} \times \mathsf{X} \times \mathsf{Y}$ as Z. A transformation g: $\mathsf{Z} \rightarrow \mathsf{Z}$ with $g(p,x,y) = (q,u,v)$ [$q \in \mathsf{P}$, $u \in \mathsf{X}$, $v \in \mathsf{Y}$] is said to *leave* Σ *invariant* if $\Sigma(q(p,x,y), u(p,x,y), v(p,x,y))$ is equivalent to $\Sigma(p,x,y)$, i.e., the two systems have identical form. The transformation g will be called an invariant transformation. To ensure that parameters remain parameters (independent of the variables) after the transformation, q is required to be dependent on p only. Therefore, (q,u,v) can be written as $(g_1(p), g_2(p, x, y), g_3(p, x, y))$. Invariant transformations will be assumed to be of this form.

We shall assume that for any fixed $p \in \mathsf{P}$, $\Sigma(p,x,y)$ has a *unique* solution f_p: $\mathsf{X} \rightarrow \mathsf{Y}$. When p changes, a set of solutions

$$\mathcal{F} \equiv \{ f_p | f_p \quad \text{satisfies} \quad \Sigma(p,x,y) \text{ for } p \in \mathsf{P} \}$$

is obtained. We say that the solutions $f_p, f_q \in \mathcal{F}$ are similar ($f_p \sim f_q$) if $\exists$ (there exists) a one–one, onto transformation h: $\mathsf{X} \times \mathsf{Y} \rightarrow \mathsf{X} \times \mathsf{Y}$ such that

$$(x,y) \in f_p \qquad \text{if } h(x,y) \in f_q$$

(f_p and f_q are treated as binary relations in the above expressions). h is called a similarity transformation. The relation defined above is an equivalence relationship in $\mathcal{F}$ (see Birkhoff and MacLane, 1965). Therefore, a quotient set $\mathcal{F}/\sim$ can be formed to contain dissimilar classes of solutions generated by P; within each class, all solutions can be generated by one known solution in the same class through similarity transformations. This is exactly what we want to match the common sense of "similarity." The quotient set turns out to be a nice device to categorize the related solutions. The knowledge of one solution per class would suffice for a knowledge of all solutions.

3. THEOREMS RELATING GROUP OF INVARIANT TRANSFORMATIONS TO SIMILARITIES AMONG SOLUTIONS

In the previous section the problem of similarity has been identified with the problem of finding equivalence classes of solutions in $\mathfrak{F}$. The subsets of parameters corresponding to these classes ($p, q \in$ the same subset if $f_p, f_q \in$ the same class; the collection of these subsets will be labeled as $\mathcal{C}$) contain parameters which can be "shrunk" to an arbitrary "representative" parameter in each subset and still generate the same class of solutions. The possibility of such "shrinking" is the actual meaning of reduction of parameters. More precisely, consider the function $\mathbb{S}: \mathsf{P} \to \mathfrak{F}$ defined as

$$\mathbb{S}(p) = f_p$$

which maps each parameter to its corresponding solution. It is well defined as f_p is unique and is onto by the definition of $\mathfrak{F}$. It may or may not be continuous (the topology of $\mathfrak{F}$ can be chosen to be the relative topology of the whole function space). The quotient $\mathfrak{F}/\sim$ is also a topological space with the quotient map

$$Q_1: \mathfrak{F} \to \mathfrak{F}/\sim$$

continuous. The composition map $Q_1 \circ \mathbb{S}: \mathsf{P} \to \mathfrak{F}/\sim$ is onto (and continuous if $\mathbb{S}$ is continuous) and can be viewed as the generation of dissimilar classes by parameters.

Definition. The range of parameters is said to be *reducible* if $\exists$ a *proper* subset P' of P such that $Q_1 \circ \mathbb{S}|\mathsf{P}'$ (restriction to P') is onto $\mathfrak{F}/\sim$.

The meaning of this definition is quite obvious: if the same amount of dissimilar solutions can be generated by a smaller range of parameters, this smaller set can be used without losing information.

The next question is how to find these equivalence classes. We want to show how the group method (see Birkhoff, 1960; Bluman and Cole, 1974) can be nicely tailored for this goal. For this purpose, two theorems are proved here. The first one, which relates the existence of a group of invariant transformations to the existence of similarities in the solutions, provides the foundation for the application of the method. The basic idea is to show that the action of the group on P traces out orbits (as subsets of P) each of which generates the same class in $\mathfrak{F}/\sim$ (i.e., each orbit is a subset of an element in $\mathcal{C}$). Restricting $Q_1 \circ \mathbb{S}$ to the subset of "representative" elements from these orbits (one from each) can generate all dissimilar classes of solutions. Of course, there may be repetition in the generation process (the restricted function, even though onto, is not one–one). The capability of the group to reduce parameters depends on the size and effectiveness of the group. Theorem 2 provides a more definitive estimate by stating that a subspace of P with dimension$=$dim(P)$-$dim(orbits) can be found to generate all dissimilar classes of solution when the group is a Lie group.

> *Theorem 1.* Let G be a group of transformations on Z, whose elements leave Σ invariant and its projective orbits in P are not all singletons; then P is reducible.

The projective orbit of G in P through p_0 is defined as

$$\mathrm{orb}(p_0)=\{p\,|\,p=g_1(p_0) \text{ for some } g\in G;\ g=(g_1,g_2,g_3)\}$$

The collection of orbits defines an equivalence relation Δ in P ($a\Delta b$ if a,b belong to the same orbit) and a quotient set P/Δ can be formed. Before we prove this theorem, let us first prove two lemmas; the first one, which relates (almost identifies) transformations leaving Σ invariant with similarity transformations, is fundamental.

> *Lemma 1 (Basic Lemma).* If g is an element of the group G which leaves Σ invariant and g_1 maps a to b in P then $f_a \sim f_b$.

Proof. That g leaves Σ invariant means that $\Sigma(b,u,v)$ or $\Sigma(g_1(a), g_2(a,x,y), g_3(a,x,y))$ is equivalent to $\Sigma(a,x,y)$. f_b is a solution of $\Sigma(b,u,v)$. Define h: $\mathsf{X}\times\mathsf{Y}\to\mathsf{X}\times\mathsf{Y}$ to be $h(x,y)=(g_2(a,x,y),g_3(a,x,y))$. [The inverse of h is just $(h^{-1})(u,v)=((g^{-1})_2(b,u,v),(g^{-1})_3(b,u,v))$.] The surface $\{(x,y)|\,(u,v)=h(x,y),(u,v)\in f_b\}$ satisfies $\Sigma(g_1(a),g_2(a,x,y),g_3(a,x,y))$ as well as $\Sigma(a,x,y)$. By assumed uniqueness, it is identical to f_a. Therefore, $(x,y)\in f_a$ if $h(x,y)\in f_b$, implying that $f_a\sim f_b$. h is the required similarity transformation. ∎

Defining Q_2 to be the quotient map from P to P/Δ, we have the following lemma.

> *Lemma 2.* The composite function $Q_1\circ\mathfrak{S}\circ Q_2^{-1}$: $\mathsf{P}/\Delta\to\mathfrak{F}/\sim$ is well defined and onto. It is continuous if $\mathfrak{S}$ is continuous.

Proof. If $a,b\in\mathrm{orb}(p_0)\in\mathsf{P}/\Delta$, then $\exists\ g\in G$ such that $g_1(a)=b$. By Lemma 1, this implies that $f_a\sim f_b$, therefore, the composite function is well defined. If U is open in $\mathfrak{F}/\sim$, by definition $Q_1^{-1}(U)$ is open (Q_1^{-1} may not be continuous, but Q_1 is open and continuous). If $\mathbb{S}$ is continuous, $\mathbb{S}^{-1}(Q_1^{-1}(U))$ is open in P and $Q_2(\mathbb{S}^{-1}(Q_1^{-1}(U))$ is open in P/Δ. ∎

Proof of Theorem 1. Select one element from each projective orbit to form a subset P', i.e., find a set $\mathsf{P}'=Q_2^{-1}(\mathsf{P}/\Delta)$ (Q_2^{-1} exists because of the axiom of choice). Lemma 2 implies that $Q_1 \circ \mathbb{S}\,|\mathsf{P}'$ from P' to $\mathcal{F}/\sim$ is onto as it can be written as $Q_1 \circ \mathbb{S} \circ Q_2^{-1} \circ Q_2|\mathsf{P}'$ and $Q_2|\mathsf{P}'$ is onto. (It is continuous in the relative topology if $\mathbb{S}$ is continuous.) P' is a proper subset of P as not all the projective orbits are singletons. ∎

The proof of Theorem 1 demonstrates the correspondence between the orbits generated by G and the equivalence classes of similar solutions in $\mathcal{F}$. The function from P' to $\mathcal{F}/\sim$, even though onto, is not necessarily one–one. "Onto" ensures that all dissimilar solutions can be generated by P', but overlapping may occur.

Theorem 1 can be applied to very general situations (e.g., when P is discrete or finite), but a more specific corollary would be more helpful for reducing the *number* of parameters (l). Consider P to be of the form

$$\mathsf{P}=\{(p_1,\ldots,p_l)|p_1\in I_1,\ldots,p_l\in I_l\} \tag{3.1}$$

where $I_i \subseteq \mathsf{R}(1 \leqslant i \leqslant l)$ are subsets of R. Each p_i is called *a* parameter; l is the *number* of parameters. We say that the *number* of parameters can be *reduced* by s if $\exists$ a subset $\mathsf{P}' \subset \mathsf{P}$ of the following form:

$$\mathsf{P}'=\{(p_1,\ldots,p_l)|p_{\alpha_1}=\text{const},\ldots,p_{\alpha_s}=\text{const}, \text{other } p_i \in I_i\} \tag{3.2}$$

such that $Q_1 \circ \mathbb{S}\,|\mathsf{P}'$ is onto $\mathcal{F}/\sim$.

> *Corollary.* Suppose that P has the form (3.1) and every element in P is in a projective orbit of G through some element belonging to a fixed subset of the form (3.2), then P is reducible to P' and the number of parameters can be reduced by s.

Proof. All orbits are through some point in P', therefore the restricted function $Q_2|\mathsf{P}'$ is onto P/Δ. Thus the composite $Q_1 \circ \mathbb{S} \circ Q_2^{-1} \circ Q_2|\mathsf{P}'$ is onto. Again if $\mathbb{S}$ is continuous, this function is continuous (when P' has the relative topology). ∎

We should remark that P' which satisfies the conditions of the above corollary is not unique; usually, there is freedom in choosing which parameters are to be eliminated. However, s can be seen to be always less than the dimension of the projective orbits. To show that the dimension of the projective orbits (σ) can be fully utilized, we shall show that s can indeed reach σ in certain new coordinate systems for P.

> *Theorem 2.* Suppose that P having form (3.1) is an analytic manifold of dimension l. If G is a Lie group whose elements leave Σ invariant and all the projective orbits under G in P have the same dimension σ, then locally $\exists$ coordinates (new parametrization) q_i $(1 \leqslant i \leqslant l)$ in which $q_{\sigma+1} = \text{const}, \ldots q_l = \text{const}$ describe sections of projective orbits and $\exists$ a subset P' of P which has dimension $l-\sigma$ (covering dimension) such that $Q_1 \circ \mathbb{S}\,|\mathsf{P}'$ is onto.

Proof (sketched). The infinitesimal transformations generated by the Lie group in P form a Lie algebra (see Chevalley, 1946). They define an integrable C^∞ σ-dimensional distribution D in P as the dimension of the tangent spaces to the orbits have the same dimension σ. Therefore, P is *foliated* by an integral manifold of D (see Spivak, 1970). Each slice of this manifold is a section of a projective orbit. Locally, $\exists$ coordinate systems of the form (U,q) where U is an open set in P with $q(U) = (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times\cdots\times(-\varepsilon,\varepsilon)$ such that sections of projective orbits in U have the form $\{p\in U | q_{\sigma+1}(p) = \text{const},\ldots,q_l(p) = \text{const}, q_1,\ldots,q_\sigma \in(-\varepsilon,\varepsilon)\}$ (ε is a real number). A countable subcollection of all these coordinate systems covers $\mathsf{P}\subset\mathsf{R}^l$ as it can be expressed as the union of a countable number of compact subsets of R^l. The union of all points p that satisfy $q_1(p) = \cdots = q_\sigma(p) = 0$ in these subcollections of coordinate systems is the required P'. Countability of the subcollection ensures that the covering dimension of the union is $l-\sigma$. ∎

Notice that the dimension of the projective orbits is not necessarily equal to the dimension of G even though it cannot be larger. G may not be *effective* on P as G may contain a nontrivial normal subgroup whose elements leave P unchanged and this subgroup reflects the possibility of reducing independent variables.

This theorem should be compared with its counterpart in ordinary similarity analysis (see p. 145, Birkhoff, 1960; p. 160, Bluman and Cole, 1974). In our proof, the usual requirement that the Lie group is solvable is not necessary.

4. DISCUSSION

When we want to test whether a certain group of transformations would leave Σ invariant we substitute the general form of the functions in Σ and derive constraints on them by requiring that the form of Σ be preserved. From the theorems proved in Section 3, one can see that this procedure is in fact equivalent to the use of extended infinitesimal transformations for testing invariance. We will not elaborate on this technique here. [This method was first developed by Sophus Lie in the 1890's, and the readers are referred to Bluman and Cole (1974) for a recent exposition of the subject.] For the purpose of making the concepts in previous sections more concrete, we discuss the application to a simple example in detail here.

Let us consider Schwarzchild's scheme (Schwarzchild, 1958) of finding solutions for the equations describing stellar structures. This scheme involves matching of integrations from surface inward and from center outward. Therefore a series of solutions is needed for the matching. As a demonstration, we only discuss outward integrations from the center. The situation is described by the following system of equations:

$$\frac{dp}{dx} = -\frac{p}{t}\frac{q}{x^2}$$

$$\frac{dq}{dx} = \frac{p}{t}x^2$$

$$\frac{dt}{dx} = -C\frac{p}{t^{6.5}}\frac{f}{x^5} \qquad \text{(radiative case)}$$

or

$$= \frac{2}{5}\frac{t}{p}\frac{dp}{dx} \qquad \text{(convective case)}$$

$$\frac{df}{dx} = Dp^2t^{\mu-2}x^2$$

with boundary conditions

$$\text{at } x=0 \qquad q=0, \qquad f=0, \qquad p=p_c, \qquad t=t_c$$

where C, D, p_c, t_c are free parameters (totalling four), μ is a fixed constant, and p, q, t, f has the physical meaning of pressure, mass, temperature, and flux, respectively. First, consider the radiative case. It is extremely tedious to find solutions for all individual values of these parameters. Fortunately, because of parametric symmetries, the number of parameters can be reduced. Substituting affine transformations of both parameters and variables in the above system and requiring that the form of the system of equations be preserved, we obtain the following group of invariant transformations:

$$\{g|g(C,D,p_c,t_c,x,p,q,t,f)=(\alpha_1 C,\ldots,\alpha_9 f) \text{ with } \alpha_7/\alpha_8\alpha_5=1,\\ \alpha_5^3\alpha_6/\alpha_7\alpha_8=1,\ \alpha_1\alpha_6^2\alpha_9/\alpha_8^{7.5}\alpha_5=1,\ \alpha_2\alpha_5^3\alpha_6^2\alpha_8^{\mu-2}/\alpha_9=1,\\ \alpha_3/\alpha_6=1,\ \alpha_4/\alpha_8=1\}$$

which has projective orbits of dimension 3. Choosing $\alpha_1=1/C$, $\alpha_2=1/D$, and $\alpha_4=1/t_c$, one can relate all integrals of the general system to integrals with only one free parameter p_c by

$$\frac{p}{p_0}=F_1\left(\frac{x}{x_0};\frac{p_c}{p_0}\right)$$

$$\frac{q}{q_0}=F_2\left(\frac{x}{x_0};\frac{p_c}{p_0}\right)$$

$$\frac{t}{t_0}=F_3\left(\frac{x}{x_0};\frac{p_c}{p_0}\right)$$

$$\frac{f}{f_0}=F_4\left(\frac{x}{x_0};\frac{p_c}{p_0}\right)$$

where $x_0\equiv 1/\alpha_5$, $p_0\equiv 1/\alpha_6$, $q_0\equiv 1/\alpha_7$, $t_0\equiv 1/\alpha_8$, $f_0\equiv 1/\alpha_9$ and $(F_1(x;p_c),\ldots,F_4(x;p_c))$ are solutions of the restricted system with $C=D=t_c=1$. As for the convective case, only D,p_c,t_c are parameters. An invariance group having the form

$$\{g|g(D,p_c,t_c,x,p,q,t,f)=(\alpha_1 D,\ldots,\alpha_8 f) \quad \text{with} \quad \alpha_6/\alpha_7\alpha_4=1,\\ \alpha_4^3\alpha_5/\alpha_6\alpha_7=1,\ \alpha_1\alpha_4^3\alpha_5^2\alpha_7^{\mu-2}/\alpha_8=1,\ \alpha_2/\alpha_5=1,\ \alpha_3/\alpha_7=1\}$$

exists. The dimension of the orbits is 3. This indicates that only *one* integration is needed to be performed for all similar solutions. From this example we can see how explicit consideration of changeable parameters can lead to a tremendous saving of effort in numerical work.

Returning to Reynold's modeling mentioned in the Introduction, we remark that to allow for invariant transformations, the boundary surface cannot be completely free. The validity of Reynold's modeling requires that the boundary surface be parametrized as $S(\mathbf{x}/L)=0$, where L is a length parameter and similarity simply means that all solutions generated by arbitrary values of ν, U_0, and L can be obtained from the subset of solutions $\{\mathbf{v}=\mathbf{F}(\mathbf{x};\nu)\}$ for the restricted cases $U_0=L=1$ (ν is the only free parameter). The above assertion can be easily verified by observing that the group of transformations $\{ g | g(U_0, L, \nu, \mathbf{x}, \mathbf{v}) = (\alpha_1 U_0, \alpha_2 L, \alpha_3 \nu, \alpha_4 \mathbf{x}, \alpha_5 \mathbf{v})$ with $\alpha_5 = \alpha_3/\alpha_4$, $\alpha_5 = \alpha_1$, $\alpha_4 = \alpha_2\}$ leaves the system invariant. Reinterpretation of dimensional analysis in this way provides a definite mathematical meaning for employing dimensional analysis.

In summary, the main result of this paper is the clarification of the concept of similarity, which was then demonstrated to be very useful in establishing the correspondence between groups of invariant transformations and similarities (Theorems 1 and 2). With an aim to the investigation of the role of parameters, we had not elaborated on the application. However, as the theorems are very general, they can be used as guides even in complicated situations.

ACKNOWLEDGMENTS

We thank T. W. Leung for discussions. This research is partially supported by an Operating Grant from the National Research Council of Canada.

REFERENCES

Birkhoff, G. (1960). *Hydrodynamics*, 2nd ed., Chap. 5. Princeton University Press, Princeton, New Jersey.

Birkhoff, G., and MacLane, S. (1965). *A Survey of Modern Algebra*. Macmillan, New York.

Bluman, G. W., and Cole, J. D. (1974). *Similarity Methods for Differential Equations*. Springer-Verlag, New York.

Chevalley, C. (1946). *Theory of Lie Groups*, Princeton University Press, Princeton, New Jersey.

Schwarzschild, M. (1958). *Structure and Evolution of Stars*, Chap. 3. Princeton University Press, Princeton, New Jersey.

Spivak, M. (1970). *A Comprehensive Introduction to Differential Geometry*, Vol. I, Chap. 6. Publish or Perish Inc., Boston, Massachusetts.